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PATH INTEGRAL BOSONIZATION OF THREE FLAVOUR QUARK DYNAMICS

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A b s t r a c t

Chiral symmetry breaking due to instanton-produced fermion zero modes in the confining vacuum is considered. Zero modes provide 'tHooft-type determinant quark interaction, which is bosonized by introduction of auxiliary fields. For three flavours this procedure becomes nontrivial and a method is suggested which allows to derive effective Lagrangian for massless Nambu–Goldstone modes.

1 Introduction

Recently effective action has been obtained for a theory with confining background superimposed on instantons [1]. This is an extension of earlier investigations of the pure instanton vacuum [2-4]. The aim of [1] and of the present letter is to formulate a model of the QCD vacuum with properties of confinement and chiral symmetry breaking (CSB) at the same time. However in [1] only one– and two– flavour ($N_f = 1, 2$) cases have been considered. It is known that bosonization becomes really nontrivial for $N_f > 2$ [5]. In the present paper we perform explicit path integral bosonization of quark dynamics outlined in [1] for $N_f = 3$ case.

First we briefly remind the basic features of the QCD model developed in [1]. The driving idea is to consider chiral and confinement effects as interconnected. To this end the QCD vacuum is assumed to consist of instantons plus confining configurations. Thus the starting point is the ansatz for the vacuum

$$A_\mu = \sum_{i=1}^{N} A^i_\mu + B_\mu \ , \quad N = N_+ + N_- \ , \tag{1.1}$$

where A^i_μ is the field of i-th (anti)instanton (N is the total number of instantons and antiinstantons in the 4–volume V), B is the background field which ensures confinement (i.e.correlators $F_{\mu\nu}(B)$ yield nonzero srtring tension). Instantons through fermion zero modes give rise to CSB.

Confining background on one hand modifies the instanton density [6], and on the other it interplays with instantons and hence modifies chiral effects [1].

As shown in [1] the gauge invariant partition function of quarks in the field (1.1) in the limit of zero quark masses can be written in a form similar to that of the pure instanton gas [4]:

$$Z = \int D\psi D\psi^+ D\mu(B) \int dg_+ dg_- expW \ , \tag{1.2}$$

$$W = -N_+ lng_+ - N_- lng_- + \int d^4u \{\psi^+_f (i\hat{D})\psi_f + g_+ detJ_+ + g_- detJ_-\} \ , \tag{1.3}$$

where the action of the gluonic field is included in $D\mu(B)$, the integration over g_+ and g_- is a remnant of the inverse Laplace transform introduced in order to rewrite the partition function in the exponential form [4], $f = 1, 2, ..., N_f$ is the flavour index, N_+ and N_- were defined in Eq.(1.1), $\hat{D} \equiv \hat{D}(B_\mu)$ is the covariant derivative corresponding to the field B_μ, the $N_f \times N_f$ matrices $J_\pm$ are related to $2N_f$ fermion vertices $Y_\pm$ via the relations

$$Y_\pm = \int d^4u detJ_\pm(u) \ , \tag{1.4}$$

$$(J_\pm(u))_{fg} = \int dxdy\psi_f^+(x)\frac{1}{2}(1\mp\gamma_5)K(x,y,u)\psi_g(y)\ , \tag{1.5}$$

$$K(x,y,u) = i\hat{D}\bar{\varphi}(x-u)\Phi(x,u,y)\bar{\varphi}^+(y-u)i\hat{D}\ , \tag{1.6}$$

where $\bar{\varphi}(x)$ is the instanton zero–mode solution:

$$\bar{\varphi}(x) = \frac{1}{\pi}\frac{\rho}{(x^2+\rho^2)^{3/2}}\quad\frac{x_\mu\gamma_\mu}{\sqrt{x^2}} \tag{1.7}$$

and $\Phi(x,u,y)\equiv\Phi(x,u)\Phi(u,y)$ is a product of parallel transporters

$$\Phi(x,u) = Pexp\ (ig\int_u^x B_\mu dz_\mu). \tag{1.8}$$

Note that $J_\pm$ and W are gauge invariant due to the presence of long derivatives $\hat{D}$ and parallel transporters in (1.6).

Expressions (1.2-1.3) for the partition function are the starting point for the rest of the paper. Our goal is to derive an effective action by integrating over the fermionic degrees of freedom. The most efficient way to do this is to introduce auxiliary fields (bosonization), then perform the integration over fermions exactly, and then apply the stationary–phase approximation to the integral over the auxiliary field. As a result one obtains the effective action for Nambu–Goldstone modes.

2 The case of two flavours

The usual path integral bosonization is based on the Hubbard–Stratonovich transformation [7]. For the quark dynamics under consideration this procedure has been performed in [4] and [1]. However it faces problems for $N_f > 2$, and more sophisticated manipulations are needed. In this section we shall reconsider the $N_f = 2$ case (see Ref.[1]) within the framework of the approach applicable for arbitrary N_f. The basics of the bosonization techniques which will be used here have been developed in [8,9].

Consider the partition function (1.2-1.3) for $N_f = 2$. The general relation, which holds for any 2×2 matrix, allows to write

$$\det J_\pm = \frac{1}{2}[(trJ_\pm)^2 - trJ_\pm^2]\ . \tag{2.1}$$

Expanding $J_\pm$ over a complete set of three Pauli matrices $\tau^a(a=1,2,3)$ plus a matrix $\tau^0 = -i\hat{1}$ yields

$$J_\pm = \sum_{a=0}^{3}\tau^a c_a^\pm\ ,\quad c_{a=1,2,3}^\pm = \frac{1}{2}tr(\tau^a J_\pm)\ ,\quad c_0^\pm = -\frac{1}{2}tr(\tau^0 J_\pm). \tag{2.2}$$

Then by virtue of (2.1) we get

$$det J_{\pm} = -\sum_{a=0}^{3}(c_a^{\pm})^2 \ . \tag{2.3}$$

Next we split out from the partition function Z given by Eqs. (1.2-1.3) the following part:

$$\tilde{Z} = \int D\psi D\psi^{+} exp\{\int d^4u[\psi_f^{+}(i\hat{D})\psi_f + g_{+}detJ_{+}(u) + g_{-}detJ_{-}(u)]\} \tag{2.4}$$
$$\equiv \int D\psi D\psi^{+} exp\tilde{W}.$$

This expression can be written in the following equivalent form (index a runs from 0 to 3):

$$\tilde{Z} = \int D\psi D\psi^{+} \prod_a D\sigma_a^{+} \prod_a D\sigma_a^{-} \delta(\sigma_a^{+} - g_{+}c_a^{+})\delta(\sigma_a^{-} - g_{-}c_a^{-}) exp(\tilde{W}) =$$
$$= \int D\psi D\psi^{+} \prod_a D\sigma_a^{+} \prod_a D\sigma_a^{-} \prod_a DM_a^{+} \prod_a DM_a^{-} exp(\tilde{W}) \times$$
$$\times \, exp\{i\int d^4u[M_a^{+}(\sigma_a^{+} - g_{+}c_a^{+}) + M_a^{-}(\sigma_a^{-} - g_{-}c_a^{-})]\} \ . \tag{2.5}$$

At the next step we integrate out in (2.5) the quark fields ψ and ψ^{+}, i.e. perform the integration

$$\tilde{Z}_{\psi} = \int D\psi D\psi^{+} exp\{i\int d^4u[\psi_f^{+}\hat{D}\psi_f - g_{+}M_a^{+}c_a^{+} - g_{-}M_a^{-}c_a^{-})]\} \ , \tag{2.6}$$

where according to (2.2) and (1.5) we have

$$M_a^{\pm}c_a^{\pm} = \frac{1}{2}\sum_a M_a^{\pm}\sum_{fg}(\tau^a)_{fg}J_{gf}^{\pm} =$$
$$= \int d^4x d^4y \psi_g^{+}(x)[\frac{1}{2}M^{\pm}(u)(\frac{1 \mp \gamma_5}{2})K(x,y,u)]\psi_f(y) \ , \tag{2.7}$$

with $M^{\pm}(u) \equiv \{M_{a>0}^{\pm}(u)\tau^{a>0}, \ -M_0^{\pm}\tau^0\}$. The integration yields:

$$\tilde{Z}_{\psi} = -exp(\ln \ detX) \tag{2.8}$$

$$X = \begin{pmatrix} -i(D_4 - i\vec{\sigma}\vec{D}) & \frac{i}{2}g_{+}M^{+}K \\ & \\ \frac{i}{2}g_{-}M^{-}K & -i(D_4 + i\vec{\sigma}\vec{D}) \end{pmatrix}. \tag{2.9}$$

Next one performs integration over the fields $\sigma_a^{\pm}$

$$\tilde{Z}_{\sigma} = \int \prod_a D\sigma_a^{+} \prod_a D\sigma_a^{-} exp\{\int d^4u[g_{+}detJ_{+} + g_{-}detJ_{-} + iM_a^{+}\sigma_a^{+} + iM_a^{-}\sigma_a^{-}]\} \ . \tag{2.10}$$

Making use of (2.3) and (2.5) one has

$$g_\pm \det J_\pm = -\frac{(\sigma_a^\pm)^2}{g_\pm} \;, \tag{2.11}$$

where summation over a is implied. Therefore integral (2.10) is Gaussian with the result

$$\tilde{Z}_\sigma = exp\{-\frac{1}{4}\int d^4u[g_+(M_a^+)^2 + g_-(M_a^-)^2]\} \;. \tag{2.12}$$

With the identification of the fields $M_a^\pm$ with L_a and R_a of Refs. [1,4]

$$L_a = \frac{i}{2}M_a^+ \;, \quad R_a = \frac{i}{2}M_a^- \;. \tag{2.13}$$

the final results for fully bosonic effective action reads

$$exp\{-S(L,R)\} = \int dg_+ dg_- exp\{-N_+ lng_+ - N_- lng_- +$$

$$+\int d^4u[g_+L_a^2(u) + g_-R_a^2(u)] + ln\ det\ X \;, \tag{2.14}$$

$$X = \begin{pmatrix} -i(D_4 - i\vec{\sigma}\vec{D}) & g_+LK \\ & \\ g_-RK & -i(D_4 + i\vec{\sigma}\vec{D}) \end{pmatrix} . \tag{2.15}$$

For further purposes it is instructive to rewrite the term $ln\ det\ X$ in (2.14-2.15) in a slightly different form. In (2.14-2.15) the determinant in Dirac space is written down explicitly. Instead let this step be implied in $det\ X$ notation. Then making use of the identity $\ln\det\ X = tr\ln X$ one can recast $\ln\det X$ in (2.14) into the form

$$\ln\det X = tr\ln\{i\hat{D} + g_+(\frac{1-\gamma_5}{2})LK + g_-(\frac{1+\gamma_5}{2})RK\} \;, \tag{2.16}$$

This form will be used in Section 4 in order to define the chiral effective Lagrangian.

We have retrieved the results obtained in [1]. The reader is referred to [1] for a discussion of the physics behind Eqs. (2.14-2.16). For $N_f = 2$ our present derivation is more cumbersome than the Hubbard–Stratonovich transformation used in [1]. However our goal was to display the algorithm applicable for $N_f > 2$ case.

3 Bosonization of Three Flavours Dynamics

Again our starting point is the partition function given by Eqs.(1.2-1.3), and our task is the integration over fermionic degrees of freedom.

For the case $N_f = 3$ relations (2.1-2.3) are replaced by

$$\det J_\pm = \frac{1}{6}(trJ_\pm)^3 - \frac{1}{2}(trJ_\pm)(trJ_\pm^2) + \frac{1}{3}trJ_\pm^3 \ , \tag{3.1}$$

$$J_\pm = \sum_{a=0}^{8} \lambda^a c_a^\pm \ , \quad c_{a>0} = \frac{1}{2}tr\lambda^a J_\pm \ , \quad c_0 = \frac{1}{3}tr(\lambda^0 J_\pm) \ , \tag{3.2}$$

$$\det J_\pm = (c_0^\pm)^3 - \frac{5}{2}c_0^\pm \sum_{a=1}^{8}(c_a^\pm)^2 + \frac{1}{3}\sum_{a,b,d=1}^{8} tr(\lambda^a\lambda^b\lambda^d)c_a^\pm c_b^\pm c_d^\pm \ . \tag{3.3}$$

Here $\lambda^a, \quad a = 1, 2, .., 8$ are the SU(3) Gell-Mann matrices, $\lambda^{a=0} = \hat{1}$.

Next one again proceeds as in equations (2.4-2.6) with the only difference that the index a now runs from 0 to 8. In (2.7) the τ_a matrices are replaced by λ^a, so that now $M^\pm(u) \equiv M_a^\pm(u)\lambda^a$. In line with (2.8-2.9) and (2.16) integration over $D\psi D\psi^+$ yields

$$\tilde{Z}_\psi = exp\{tr \ln[i\hat{D} + g_+(\frac{1-\gamma_5}{2})LK + g_-(\frac{1+\gamma_5}{2})RK]\} \ . \tag{3.4}$$

Here $L = \frac{i}{2}M^+ = L_a\lambda^a \ , \quad R = \frac{i}{2}M^- = R_a\lambda^a$, so that (3.4) contains 18 bosonic fields $L_a, R_a, a = 0, 1, ..., 8$.

The next step is the integration over $\sigma_a^\pm$ and that is where the real difference from the $N_f = 2$ case starts. The integral analogous to (2.10) reads

$$\tilde{Z}_\sigma = \int \prod_a D\sigma_a \exp\{-\int d^4u[-\frac{1}{g^2}(\sigma_0^3 - \frac{5}{2}\sigma_0\sum_{a=1}^{8}\sigma_a^2 + \frac{1}{3}\sum_{a,b,d=1}^{8} tr(\lambda^a\lambda^b\lambda^d)\sigma_a\sigma_b\sigma_d) + \tag{3.5}$$

$$-i\sum_{a=0}^{8} M_a\sigma_a]\} \equiv \int \prod_a D\sigma_a \exp\{S(\sigma_a)\}.$$

In (3.5) we have suppressed the $(\pm)$ indices, so that additional summation over $(\pm)$ is implied. The $N_f = 2$ integral (2.10) was Gaussian and therefore it could be calculated both exactly and by stationary phase method yielding the same results.Exact analytical calculation of (3.5) is not possible. Therefore one has to resort to the stationary phase, or the steepest descent methods. The corresponding conditions which define the $\sigma_a = f(\{M_a\})$ solutions have the form

$$\frac{\delta S}{\delta\sigma_0} = 3\sigma_0^2 - \frac{5}{2}\sum_{a=1}^{8}\sigma_a^2 + ig^2M_0 = 0$$

$$\frac{\delta S}{\delta\sigma_a} = -5\sigma_0\sigma_a + \frac{1}{3}\sum_{b,d=1}^{8} tr(\lambda^a\lambda^b\lambda^d)\sigma_b\sigma_d + ig^2 M_a = 0 \ , \quad a = 1,2,...,8 \ . \tag{3.6}$$

The principal question is the existence of a contour along which the integral (3.5) converges, so that the steepest descent method yields reliable results. As a toy model consider the case $\sigma_0 \neq 0$, $\sigma_a = 0$, $a = 1,2,...,8$ (for $N_f = 2$ similar situation was investigated in [1,4]). Then (3.5) reduces to

$$\tilde{\tilde{Z}}_\sigma = \int D\sigma exp\{-\int d^4u(-\frac{\sigma^3}{g^2} - iM\sigma)\} \tag{3.7}$$

with $\sigma \equiv \sigma_0$. Replacing $M = -2iL,\ g = i\kappa$, we get

$$\tilde{\tilde{Z}}_\sigma = \int D\sigma exp\{\int d^4u(2L\sigma - \frac{\sigma^3}{\kappa^2})\} \ . \tag{3.8}$$

This is an Airy type integral [10]. In the Appendix we show how to choose the correct contour and how to calculate this integral by the steepest descent method. *Up* to unimportant overall factor the result is

$$\tilde{\tilde{Z}}_\sigma = exp\{-\frac{4}{3}\sqrt{\frac{2}{3}}\kappa\int d^4uL^{3/2}(u)\} \ . \tag{3.9}$$

This result is reminisced of the stationary phase condition for Landau magnetisation function of the infinite range Ising model [7].

4 The gap equation

For the toy model under consideration the partition function (1.2-1.3) reads

$$Z \propto \int DL\int d\kappa exp\{-N\ln\kappa - \frac{4}{3}\sqrt{\frac{2}{3}}\kappa\int d^4uL^{3/2} + tr\ln(-D^2 + \kappa^2L^2K^2)\} \ . \tag{4.1}$$

The steepest descent integration yields

$$-2\sqrt{\frac{2}{3}}\kappa VL^{1/2} + 2\kappa^2L\ tr(\frac{K^2}{-D^2+\kappa^2L^2K^2}) = 0 \ ,$$

$$-\frac{N}{\kappa} - \frac{4}{3}\sqrt{\frac{2}{3}}VL^{3/2} + 2\kappa L^2 tr(\frac{K^2}{-D^2+\kappa^2L^2K^2}) = 0 \ . \tag{4.2}$$

From (4.2) we find

$$\kappa^2L^2tr(\frac{K^2}{-D^2+\kappa^2L^2K^2}) = \frac{3}{2}N \ . \tag{4.3}$$

With the identification for the chiral mass operator $m \equiv \kappa LK$, and performing on the l.h.s. of (4.3) trivial summation over flavour indices one gets the same form of the gap equation as for $N_f = 1, 2$ in [1]

$$tr(\frac{m^2}{-D^2 + m^2}) = \frac{N}{2} \ . \tag{4.4}$$

5 The effective chiral lagrangian

We turn now to the final topic, namely to the constuction of the effective chiral Lagrangian for $N_f = 3$. Such a Lagrangian corresponds to an octet of Goldstone bosons $\pi^0, \pi^\pm, K^\pm, K^0, \bar{K}^0, \eta$ which are coupled by the fermionic loop. The starting point is Eq.(3.4) in which we parametrize bosonic fields L and R in the following form [4]

$$L = L_a\lambda^a = i\rho(\frac{1+\xi+\mu}{2})UV, \ R = R_a\lambda^a = i\rho(\frac{1+\xi+\mu}{2})VU^+, \ a = 0, 1, ..., 8, \tag{5.1}$$

$$U = \exp\{\frac{1}{2}i\pi_b\lambda^b\} \ , V = exp\{\frac{1}{2}i\varepsilon_b\lambda^b\} \ , \quad b = 1, 2, ..., 8. \tag{5.2}$$

This is a general parametrization which expresses 18 bosonic fields L_a, R_a in terms of 18 fields $\xi, \mu, \pi_b, \varepsilon_b$(note that in (5.1) summation is from 0 to 8, while in (5.2) from 1 to 8). Insertion of (5.1)-(5.2) into (3.4) yields

$$\tilde{Z}_\psi = \exp\{tr \ln[i\hat{D} + i(1+\xi+\mu)UVE^+ + i(1+\xi-\mu)VU^+E^-]\} \ , \tag{5.3}$$

where $E^\pm = E(\frac{1\mp\gamma_5}{2})$, $\ E = \frac{1}{2}g\rho K$, and the operator K is defined by Eq.(1.6). In order to get the effective chiral Lagrangian one has to integrate (5.3) over all meson fields exept for the Goldstone octet π_b. Using the steepest descent in the vicinity of the vacuum sadde–point [1,4] we get

$$S_{eff}(\pi_b) = -\ln \int D\xi D\mu D\varepsilon \tilde{Z}_\psi =$$

$$= tr \ln(i\hat{D} + iEU_5) \ , \tag{5.4}$$

where

$$U_5 = U(\frac{1-\gamma_5}{2}) + U^+(\frac{1+\gamma_5}{2}) = \exp\{\frac{-i1}{2}\pi_a\lambda^a\gamma_5\} \ . \tag{5.5}$$

The form of the effective action (5.4) coincides with that found in [1] for $N_f = 2$. In absence of background gluonic field B_μ the form (5.4) reduces to the effective action found in [4] for the pure instanton case (see also discussion in [11]).

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The final note is that the authors became aware of Ref. [5] after completing the present paper. Although our approach differs from that of [5], the results we got for $N_f = 3$ fit into the sheme outlined in [5] for arbitrary N_f.

APPENDIX

In the Appendix we show how the result (3.9) emerges. Consider the integral (3.8):

$$\tilde{\tilde{Z}} = \int D\sigma exp\{\int d^4u(2L\sigma - \frac{\sigma^3}{\kappa^2})\} \ . \tag{A.1}$$

We introduce a new variable w and a new parameter t according to

$$w = \left(\frac{2\kappa^2 L}{3}\right)^{-1/2} \sigma \ , \quad t = 2\left(\frac{2}{3}\right)^{1/2} \kappa L^{3/2} \ . \tag{A.2}$$

Then (A.1) takes the form (up to a Jacobian which is not essential for our present purposes)

$$\tilde{\tilde{Z}} = \int Dwexp\{\int d^4ut(w - \frac{w^3}{3})\} \ , \tag{A.3}$$

which is the standard representation of the Airy function [10]. The leading contribution (3.9) is then obtained by the steepest descent method. The integral has two stationary points $w = \pm 1$, from which only $w = -1$ contributes. The integration contour can be choosen along the line $Rew = -1$.

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